Metric and Topological Spaces Lecture Notes (2024/2025)

Griffin Reimerink

Contents

1 Sets and maps

Definition *Set*

A **set** is a boolean function χ which sends an object to $\{0,1\}$:

$$\chi_A(a) = 1 \implies a \in A \qquad \chi_A(a) = 0 \implies a \notin A \qquad A \subseteq B \text{ if } \chi_A \leq \chi_B$$

Proposition *De Morgan's Laws*

$$S \setminus \bigcup_{i\in\mathcal{I}} A_i = \bigcap_{i\in\mathcal{I}} (S \setminus A_i) \qquad S \setminus \bigcup_{i\in\mathcal{I}} A_i = \bigcap_{i\in\mathcal{I}} (S \setminus A_i)$$

Definition *Properties of maps*

A map $f: X \to Y$ is:

- **injective** or **into** if $\quad f(x) = f(x') \implies x = x'$
- **surjective** or **onto** if $\quad \forall y \in Y\, \exists x \in X \mid f(x) = y$
- **bijective** if injective and surjective.

Notation: $f|A$ is a function f restricted to a domain A.

Definition *Image*

Let $f: X \to Y$ with $A \subseteq X$ and $C \subseteq Y$.

- The **direct image** $f(A)$ is $\{y \in Y \mid \exists a \in A \mid f(a) = y\}$
- The **inverse image** (or **pre-image**) $f^{-1}(C)$ is $\{x \in X \mid f(x) \in C\} \subseteq X$

Proposition

Let $f: X \to Y$ with $A, B \subseteq X$ and $C, D \subseteq Y$.

$$f(A \cup B) = f(A) \cup f(B) \qquad f(A \cap B) \subseteq f(A) \cap f(B)$$

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D) \qquad f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$$

Proposition

Consider the map $f: X \to Y$ and subsets A_i and C_i of X and Y respectively for all $i \in I$.

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i) \qquad f\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} f(A_i)$$

$$f^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f^{-1}(A_i) \qquad f^{-1}\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} f^{-1}(A_i)$$

Proposition

Suppose that $f: X \to Y$ is a map and $B \subset X, D \subset Y$. Then,

$$f(X \setminus B) \subset f(X) \setminus f(B) \qquad f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)$$

Proposition

Let X, Y be sets and $f: X \to Y$ a map.

- For any $C \subset Y$ we have $f(f^{-1}(C)) = C \cap f(X)$.
- For any $C \subset Y$, if f is surjective, $f(f^{-1}(C)) = C$
- For any $A \subset X$ we have $A \subset f^{-1}(f(A))$

Definition *Invertible map*

A map $f: X \to Y$ is **invertible** if there exists a map $g: Y \to X$ such that $g \circ f$ is the identity map of X and $f \circ g$ is the identity map of Y.

Proposition

A map is invertible if and only if it is bijective.

2 Real analysis

2.1 Bounds

Definition *Bounds*

$S \subseteq \mathbb{R}$ is **bounded above** if there exists an **upper bound** $u \in \mathbb{R}$ such that $x < u$ for all $x \in S$.
If S is bounded above, we call u a **least upper bound** of **supremum** for S, denoted $\sup(S)$, if:

1. u is an upper bound for S
2. $x \geq u$ for any upper bound x for S

$S \subseteq \mathbb{R}$ is **bounded below** if there exists a **lower bound** $\ell \in \mathbb{R}$ such that $x > \ell$ for all $x \in S$.
If S is bounded below, we call ℓ a **greatest lower bound** of **infimum** for S, denoted $\inf(S)$, if:

1. ℓ is a lower bound for S
2. $x \leq \ell$ for any lower bound x for S

Proposition *Completeness property*

Any non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.
Any non-empty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.

Proposition

The set $\mathbb{N}$ of positive integers is not bounded above.

Proposition *Denseness of $\mathbb{Q}$ in $\mathbb{R}$*

Between any two distinct real numbers x and y there is a rational number.
Between any two distinct real numbers there is also an irrational number.

Proposition *Triangle inequality, reverse triangle inequality*

For all $x, y \in \mathbb{R}$:

$$|x + y| \leq |x| + |y| \qquad |x - y| \geq ||x| - |y||$$

2.2 Sequences

Definition *Convergent sequence*

A sequence (s_n) **converges** to $L \in \mathbb{R}$ if:

$$\text{for all } \varepsilon > 0 \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } |s_n - L| < \varepsilon \text{ for all } n \geq N_\varepsilon$$

Proposition

A convergent sequence has a unique limit.

Proposition

Suppose that there exists $K \in \mathbb{N}$ (independent of ε) such that:

$$\text{for all } \varepsilon > 0 \text{ there exists } N_\varepsilon \text{ such that } |s_n - L| < K\varepsilon \text{ for all } n \geq N_\varepsilon$$

Then (s_n) converges to L.

Definition *Monotonic sequence*

A sequence is:

- **monotonic increasing** if $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$
- **monotonic decreasing** if $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$
- **monotonic** if it is monotonic decreasing or monotonic increasing

Proposition

Every bounded monotonic sequence of real numbers converges.

Definition *Cauchy sequence*

A sequence is a **Cauchy sequence** if:

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that if } m, n \geq N \text{ then } |s_m - s_n| < \varepsilon$$

Proposition *Cauchy's convergence criterion*

A sequence (s_n) of real numbers converges if and only if it is a Cauchy sequence.

Proposition

Every bounded sequence of real numbers has at least one convergent subsequence.

Proposition

Suppose that (s_n), (t_n) converge to s, t respectively. Then,

- $(s_n + t_n)$ converges to $s + t$
- $(s_n t_n)$ converges to st
- $1/t_n$ converges to $1/t$ provided $t \neq 0$

2.3 Limits of functions

Definition *Limit*

$f(x)$ has **limit** L at a (notation: $\lim_{x \to a} = L$) if:

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$$

$f(x)$ has **right-hand limit** L at a (notation: $\lim_{x \to a^+} = L$) if:

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } |f(x) - L| < \varepsilon \text{ for all } x \in (a, a + \delta)$$

Proposition

The following are equivalent:

1. $\lim_{x \to a} f(x) = L$
2. if (x_n) is any sequence such that (x_n) converges to a but for all n we have $X_n \neq a$, then $(f(x_n))$ converges to L.

Definition *Continuity*

A function is **continuous** at a if $\lim_{x \to a}$ exists and is equal to $f(a)$.

Proposition

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ and that $f(a) \neq 0$.
Then there exists $\delta > 0$ such that $f(x) \neq 0$ whenever $|x - a| < \delta$.

Proposition

Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous at $a \in \mathbb{R}$.
Then $f + g$, $f \cdot g$, $|f|$ and $\frac{1}{g}$ (only if $g(a) \neq 0$) are also continuous at $a \in \mathbb{R}$.

Proposition

Suppose that f is continuous at a and g is continuous at $f(a)$. Then $g \circ f$ is continuous at a.

Definition *Intermediate value property*

A function $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property (IVP) if given any a, b, d in $\mathbb{R}$ with $a < b$ and d between $f(a)$ and $f(b)$, there exists at least one c satisfying $a \leq c \leq b$ and $f(c) = d$.

Proposition

All continuous functions $\mathbb{R} \to \mathbb{R}$ (or $I \to \mathbb{R}$ where I is an interval) have the intermediate value property.

3 Metric spaces

Definition *Metric space*

A **metric space** consists of a non-empty set X together with a function $d : X \times X \to \mathbb{R}$ such that the following axioms hold:

1. **Nonnegativity** For all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
2. **Symmetry** For all $x, y \in X$, $d(x, y) = d(y, x)$
3. **Triangle inequality** For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

Notation: $\mathbb{E}^n$ is the metric space $(\mathbb{R}^n, d)$ where d is the Euclidean metric.

3.1 Continuity

Definition *Continuity of multivariable functions*

A function $f : \mathbb{R}^n \to \mathbb{R}$ is **continuous** at a point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for all $x = (x_1, \ldots, x_n)$ satisfying

$$\sqrt{\left[\sum_{i=1}^{n} (x_i - a_i)^2\right]} < \delta$$

Definition *Continuity of metric space maps*

Suppose that (X, d_X) and (Y, d_Y) are metric spaces and let $f : X \to Y$ be a map.

- f is **continuous at** $x_0 \in X$ if:
  for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$, for all $x \in X$.
- f is **continuous** (or (d_x, d_y)-continuous) if f is continuous at every $x_0 \in X$.

Proposition

Suppose that $f, g : X \to \mathbb{R}$ are continuous, real-valued functions on a metric space (X, d).
Then $f + g$, $f \cdot g$, $|f|$ and $\frac{1}{g}$ (only if $g(x) \neq 0 \quad \forall x \in X$) are also continuous.

Proposition

Suppose that $f : X \to Y$ and $g : Y \to Z$ are maps of metric spaces,
with f continuous at $a \in X$ and g continuous at $f(a)$. Then $g \circ f$ is continuous at a.

Proposition

Suppose that $f : X \to X'$, $g : Y \to Y'$ are maps of metric spaces continuous at $a \in X, b \in Y$ respectively. Then the map $f \times g : X \times Y \to X' \times Y', (x, y) \mapsto (f(x), g(y))$ is continuous at (a, b).

Proposition

The projections $p_x : X \times Y \to X$, $p_y : X \times Y \to Y$ of a metric product onto its factors,
defined by $p_x(x, y) = x$, $p_y(x, y) = y$, are continuous.

Proposition

The diagonal map $\Delta : X \to (X, X), (x) \mapsto (x, x)$ of any metric space is continuous.

3.2 Bounded sets

Definition *Bounded subset*

A subset S of a metric space (X, d) is **bounded** if there exist $x_0 \in X$, $K \in \mathbb{R}$ such that $d(x, x_0) \leq K$ for all $x \in S$.

Definition *Diameter*

If S is a non-empty bounded subset of a metric space with metric d, then the **diameter** of S is:

$$\text{If } S \text{ nonempty: } \operatorname{diam}(S) := \sup\{d(x,y) : x, y \in S\} \qquad \operatorname{diam}(\varnothing) = 0$$

Definition *Bounded function*

If $f : S \to X$ is a map from a set S to a metric space X, then we say f is **bounded** if $f(S)$ is bounded.

Proposition

The union of any finite number of bounded subsets of a metric space is bounded.

3.3 Open sets

Definition *Open ball*

Let (X, d_x) be a metric space, $x_0 \in X$ and $r > 0 (r \in \mathbb{R})$.
The **open ball** in X of radius r centered at x_0 is the set

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

If we also consider metrics other than d, we use the notation $B_r^d(x_0)$.

Proposition

$$f \text{ continuous at } x_0 \iff \forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that } f(B_\delta^{d_X}(x_0)) \subseteq B_\varepsilon^{d_Y}(f(x_0))$$

Proposition

Let $B_r(x)$ be an open ball and $y \in B_r(x)$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq B_r(x)$

Definition *Open set*

Let (X, d) be a metric space and $U \subseteq X$.
We say that U is **open** in X if for every $x \in U$ there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subseteq U$.

Proposition

Let $f : X \to Y$ be a map between metric spaces.

$$f \text{ continuous} \iff f^{-1} \text{ is open in } X \text{ whenever } U \text{ is open in } Y$$

Proposition

The intersection of a finite number of open sets in a metric space is open.
The union of any number of open sets in a metric space is open.

3.4 Closed sets and closure

Definition *Closed set*

A subset V of a metric space X is closed in X if $X \setminus V$ is open in X.

Proposition

The intersection of any number of closed sets in a metric space is closed.
The union of a finite number of closed sets in a metric space is closed.

Proposition

For any metric space X, the empty set $\varnothing$ and the whole set X are both open and closed in X.

***Definition** Closure*

Let X be a metric space and $A \subset X$.

- $x \in X$ is a **point of closure** of A in X if for all $\varepsilon > 0$ we have $B_\varepsilon(x) \cap A \neq \varnothing$.
- The **closure** of A in X (denoted $\overline{A}$) is the set of all points of closure of A in X.

***Definition** Dense subset*

A subset A of a metric space X is said to be **dense** in X if $\overline{A} = X$.

***Proposition** Properties of closure*

Let A, B be subsets of a metric space X. Then,

1. $A \subseteq \overline{A}$
2. $A \subseteq B \implies \overline{A} \subseteq \overline{B}$
3. A is closed in $X \iff \overline{A} = A$
4. $\overline{\overline{A}} = \overline{A}$
5. $\overline{A}$ is closed in X
6. $\overline{A}$ is the smallest closed subset of X containing A

Proposition

Let $f : X \to Y$ be a map of metric spaces. Then

$$f \text{ continuous} \iff f(\overline{A}) \subseteq \overline{f(A)} \quad \text{for all } A \subseteq X$$

***Definition** Limit point*

A point x in a metric space X is a **limit point** of $A \subseteq X$ if:

$$\text{for all } \varepsilon > 0 \text{ there exists a point in } B_\varepsilon(x) \cap A \text{ other than } x \text{ itself}$$

Proposition

Let A be a subset of a metric space X.

$$A \text{ is closed in } X \iff A \text{ contains all of its limit points in } X$$

Proposition

Let A be any subset of a metric space X. Then A is the union of A with all its limit points in X.

3.5 Interior and boundary

***Definition** Interior*

The **interior** $\mathring{A}$ of A is the set of points $a \in A$ such that $B_\varepsilon(a) \subseteq A$ for some $\varepsilon > 0$.

Proposition

Let A, B be subsets of a metric space X. Then,

1. $\mathring{A} \subseteq A$
2. $A \subseteq B \implies \mathring{A} \subseteq \mathring{B}$
3. A is open in $X \iff \mathring{A} = A$
4. $\mathring{\mathring{A}} = \mathring{A}$
5. $\mathring{A}$ is open in X
6. $\mathring{A}$ is the largest open subset of X contained in A.

Definition *Boundary*

The boundary ∂A of a subset A in a metric space X is the set $\overline{A} \setminus \mathring{A}$

Proposition

Let A be a subset of a metric space X and $x \in A$.

$$x \in \partial A \iff A \cap B_\varepsilon(x) \text{ and } (X \setminus A) \cap B_\varepsilon(x) \text{ are both non-empty}$$

3.6 Equivalent metrics

Definition *Topological equivalence*

Metrics d_1, d_2 on a set X are **topologically equivalent** if:

$$U \text{ is } d_1\text{-open in } X \iff U \text{ is } d_2\text{-open in } X \qquad \text{for all } U \subseteq X$$

Proposition

Suppose that d_1, d_2 are equivalent metrics for X.
Consider metric spaces (Y, d_Y), (Z, d_Z) and functions $f : Y \to Z$, $g : X \to Z$. Then,

$$f\ (d_Y, d_1)\text{-continuous} \iff f\ (d_Y, d_2)\text{-continuous} \qquad g\ (d_1, d_Z)\text{-continuous} \iff g\ (d_2, d_Z)\text{-continuous}$$

Definition *Lipschitz equivalence*

Two metrics d_1, d_2 on a set X are **Lipschitz equivalent** if there exist $h, k > 0$ such that for any $x, y \in X$:

$$hd_2(x, y) \leq d_1(x, y) \leq kd_2(x, y)$$

Proposition

Lipschitz equivalent metrics are topologically equivalent.

Definition *Lipschitz equivalence*

A **Lipschitz equivalence** is a bijective map $f : X \to Y$ where there exist $h, k > 0$ such that for any $x_1, x_2 \in X$:

$$hd_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2) \leq kd_Y(f(x_1), f(x_2))$$

Definition *Isometry*

An isometry $f : X \to Y$ is a bijective map such that for all $x_1, x_2 \in X$:

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

4 Topological spaces

Definition *Topological space*

A **topological space** $T = (X, \mathcal{T})$ consists of a non-empty set X
together with a fixed family $\mathcal{T}$ of subsets of X satisfying:

1. $\varnothing$ and X are in $\mathcal{T}$
2. the **intersection** of any finite collection of sets in $\mathcal{T}$ is in $\mathcal{T}$
3. the **union** of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$

We call $\mathcal{T}$ a **topology** for X, and if $U \in \mathcal{T}$ we say that U is **open** in $\mathcal{T}$. (or open in X)

Definition *Examples of topological spaces*

Let X be a nonempty set.
The **discrete topology** on X consists of all subsets on X.
The **indiscrete topology** on X is the family $\{\varnothing, X\}$.
The **co-finite topology** on X consists of the empty set and all subsets U of X such that $X \setminus U$ is finite.

Definition *Metrizability*

A topological space $(X, \mathcal{T})$ is **metrizable** if there exists a metric d_X such that:

$$U \in \mathcal{T} \iff U \text{ is } d_X\text{-open in } X$$

If a topological space $(X, \mathcal{T})$ is metrizable with metric d_X, we say that $\mathcal{T}$ is **induced by** d_X

Definition *Coarse and fine topologies*

Given topologies $\mathcal{T}_1, \mathcal{T}_2$ on the same set X,
we say $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$

Proposition

Let $U \subseteq X$ be a subset of a topological space.

$$U \text{ is open in } X \iff \text{for all } x \in U, \text{ there is an open subset } U_x \subseteq X \text{ such that } x \in U_x$$

4.1 Continuity

Definition *Continuity of topological spaces*

Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a map.
f is **continuous** if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.
f is **continuous at** $x_0 \in X$ if for all $U' \in \mathcal{T}_Y$ s.t. $f(x_0) \in U'$, there exists $U \in \mathcal{T}_X$ s.t. $x \in U$ and $f(U) \subseteq U'$

Proposition

$$f : X \to Y \text{ is continuous} \iff f \text{ is continuous at } x \quad \forall x \in X$$

Proposition

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces induced by metric spaces (X, d_X), (Y, d_Y) respectively.

$$f : X \to Y \text{ is } (d_X, d_Y)\text{-continuous} \iff : X \to Y \text{ is } (\mathcal{T}_X, \mathcal{T}_Y)\text{-continuous}$$

Proposition

The composition of two continuous maps $f : X \to Y$ and $g : Y \to Z$ is continuous.

Proposition

The following maps are continuous:

1. The identity map of any topological space.
2. Any constant map.
3. If $\mathcal{T}_X$ is the discrete topology, any map $X \to Y$.
4. If $\mathcal{T}_Y$ is the indiscrete topology, any map $X \to Y$.

Definition *Homeomorphism*

A **homeomorphism** between topological spaces X and Y is a bijective map $f : X \to Y$
such that f and its inverse function f^{-1} are both continuous.

4.2 Bases

Definition *Basis*

A **basis** for a topology $\mathcal{T}$ is a subfamily $\mathcal{B} \subseteq \mathcal{T}$ such that every set in $\mathcal{T}$ is a union of sets from $\mathcal{B}$.

Proposition

A map $f : X \to Y$ is continuous if for each open set B in some basis for $\mathcal{T}_Y$, the inverse image $f^{-1}(B)$ is open in X.

4.3 Closure and interior

Definition *Closed set*

A subset V of a topological space X is **closed** in X if $X \setminus V$ is open in X.

Proposition

Let X be a topological space. Then

1. $\varnothing$ and X are closed in X
2. the **intersection** of any collection of closed sets in X is closed in X
3. the **union** of any finite collection of closed sets in X is closed in X

Proposition

Let $f : X \to Y$ be a map of topological spaces.

$$f \text{ continuous} \iff f^{-1}(V) \text{ is closed in } X \text{ whenever } V \text{ is closed in } Y$$

Definition *Closure*

Let A be a subset of a topological space X.
A point a is a **point of closure** of a $A \subseteq X$ if $U \cap A = \varnothing$ for any open subset $U \subseteq X$ with $a \in U$.
The **closure** $\overline{A}$ of A in X is the set of points of closure of A in X.

Definition *Dense subset*

A subset A of a space X is **dense** in X if $\overline{A} = X$.

Proposition

Let $f : X \to Y$ be a map of topological spaces. Then

$$f \text{ continuous} \iff f(\overline{A}) \subseteq \overline{f(A)} \quad \text{for all } A \subseteq X$$

Definition *Interior*

Let A be a subset of a topological space A.
A point a is an **interior point** of $A \subseteq X$ if there exists some set U which is open in X and with $a \in U \subseteq A$.
The set of all interior points of A is called the **interior** of A, denoted $\mathring{A}$.

Proposition

We have $\overline{X \setminus A} = X \setminus \mathring{A}$ for any subset A of a space X.

Proposition *Properties of interior and closure*

Let A, B be subsets of a topological space X. Then,

1. $A \subseteq \overline{A}$
2. $A \subseteq B \implies \overline{A} \subseteq \overline{B}$
3. A is closed in $X \iff \overline{A} = A$
4. $\overline{\overline{A}} = \overline{A}$
5. $\overline{A}$ is closed in X
6. $\overline{A}$ is the smallest closed subset of X containing A
7. $\mathring{A} \subseteq A$
8. $A \subseteq B \implies \mathring{A} \subseteq \mathring{B}$
9. A is open in $X \iff \mathring{A} = A$
10. $\mathring{\mathring{A}} = \mathring{A}$
11. $\mathring{A}$ is open in X
12. $\mathring{A}$ is the largest open subset of X contained in A.

Definition *Boundary*

The **boundary** ∂A of a subset A of a space X is the set $\overline{A} \setminus \mathring{A}$.

Proposition

For any subset A of a space X, we have $\partial A = \overline{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$

Definition *Neighborhood*

A **neighbourhood** of a point x in a space X is a subset N of X which contains an open subset of X containing x.

4.4 Subspaces

Definition *Subspace topology*

Let $(X, \mathcal{T})$ be a topological space and let A be a non-empty subset of X. The **subspace topology** on A is:

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$$

Proposition

Let $(X, \mathcal{T})$ be a topological space and let A be a non- empty subset of X with the subspace topology $\mathcal{T}_A$.
Then the inclusion map $i : A \to X$ defined by $i(a) = a$ for all $a \in A$, is $(\mathcal{T}_A, \mathcal{T})$-continuous.

Proposition

Let $f : X \to Y$ be a continuous map of topological spaces $(X, \mathcal{T}), (Y, \mathcal{T}')$ and let A be a non-empty subset of X with the subspace topology $\mathcal{T}_A$. Then the restriction $f|_A : A \to Y$ is $(\mathcal{T}_A, \mathcal{T}')$-continuous.

Proposition

Let X be a topological space, let A be a subspace of X and let $i : A \to X$ be the inclusion map.
Suppose that Z is a topological space and that $g : Z \to A$ is a map. Then:

$$g \text{ is continuous} \iff i \circ g : Z \to X \text{ is continuous}$$

The subspace topology T_A on A is the only topology satisfying this proposition for all possible maps g.

4.5 Product spaces

Definition

Suppose that $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ are topological spaces.
The **product topology** $\mathcal{T}_{X\times Y}$ on $X \times Y$ is the family of all unions of sets of the form $U \times V$ where $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$.

Proposition

The following **projection maps** are continuous:

$$p_X : X \times Y \to X : (x, y) \mapsto x \qquad p_Y : X \times Y \to Y : (x, y) \mapsto y$$

Proposition

Let X, Y, Z be topological spaces and $f : Z \to X \times Y$.

$$f \text{ is continuous} \iff p_X \circ f : Z \to X \text{ and } p_Y \circ f : Z \to Y \text{ are continuous}$$

Proposition

If $f : X \to X'$, $g : Y \to Y'$ are continuous, then so is $f \times g : X \times Y \to X' \times Y'$ defined by $(f \times g)(x, y) = (f(x), g(y))$.

Proposition

For any topological space X, let $\Delta : X \to X \times X$ be the **diagonal map** defined by $\Delta(x) = (x, x)$.
The diagonal map is continuous.

Proposition

Let X and Y be topological spaces, and let $y_0 \in Y$. Define $i_{y_0} : X \to X \times Y$ by $i_{y_0}(x) = (x, y_0)$.
This map is continuous.

Proposition

If $f, g : X \to \mathbb{R}$ are continuous real-valued functions on a topological space X, then so are:

$$|f| \qquad f + g \qquad fg \qquad \frac{1}{g} \text{ if } g \text{ is never 0 on } X$$

Definition *Graph*

Let $f : X \to Y$ be a map of topological spaces. The **graph** of f is defined by:

$$G_f := \{(x, y) \in X \times Y : f(x) = y\}$$

with the topology induced by the product topology on $X \times Y$.

Proposition

Let $f : X \to Y$ be a continuous map of topological spaces.
Then the map $\theta : x \mapsto (x, f(x))$ defines a homeomorphism from X to G_f.

Proposition

$$W \subseteq X \times Y \text{ open in } X \times Y \iff \text{for all } (x, y) \in W \text{ there exist } U \subseteq \mathcal{T}_X, V \subseteq \mathcal{T}_Y \text{ with } (x, y) \in U \times V \subseteq W$$

4.6 The Hausdorff condition

Definition *Convergent sequence*

A sequence (x_n) of points in a topological space X **converges** to a point $x \in X$ if

$$\text{for all open sets } U \text{ containing } x, \text{ there exists } n \in \mathbb{N} \text{ such that } \quad n \geq N \implies x_n \in U$$

Definition *Hausdorff condition*

A topological space X satisfies the **Hausdorff condition** if

$$\text{for any two distinct points } x, y \in X \text{ there exist disjoint open sets } U, V \text{ of } X \text{ such that } x \in U, y \in V$$

We refer to a topological space which satisfies the Hausdorff condition as a Hausdorff space.

Definition *Topological property*

A **topological property** is a property of a topological space that is invariant under homeomorphisms.

Proposition

Any metrizable space $(X, \mathcal{T})$ is Hausdorff.

Proposition

1. Any subspace of a Hausdorff space is Hausdorff.
2. The topological product $X \times Y$ of spaces X and Y is Hausdorff $\iff$ both X and Y are Hausdorff.
3. If $f : X \to Y$ is an injective continuous map of topological spaces and Y is Hausdorff then so is X.
4. Hausdorffness is a topological property.

Definition *Regular and normal space*

A topological space X is **regular** if given any closed subset $V \subseteq X$ and point $x \in X \setminus V$, there exist disjoint open subsets $U, U' \subseteq X$ such that $V \subseteq U$ and $x \in U'$.
A topological space X is **normal** if given any closed subset $V \subseteq X$, and closed subset $V' \subseteq X$ disjoint from V, there exist disjoint open subsets $U, U' \subseteq X$ such that $V \subseteq U$ and $V' \subseteq U'$.

4.7 Connected spaces

Definition *Connected space*

A topological space X is **connected** if there does not exist a continuous map from X onto a two-point discrete space.

Definition *Partition*

A **partition** $A \sqcup B$ is a pair of non-empty subsets $A, B \subseteq X$ such that:

$$X = A \cup B \qquad A \cap B = \varnothing \qquad A, B \text{ open in } X$$

Proposition

Let X be a topological space. The following are equivalent:

1. X is connected.
2. There exists no partition of X.
3. The only subsets of X which are both open and closed are X and $\varnothing$.

Definition *Connected subset*

A non-empty subset A of a topological space X is connected if A with the subspace topology is connected.
The empty set is connected.

Proposition

A non-empty subset $S \subseteq \mathbb{R}$ is an interval if and only if it satisfies the following property:
if $x, y \in S$ and $z \in \mathbb{R}$ are such that $x < z < y$, then $z \in S$.

Proposition

Let S be a subspace of $\mathbb{R}$.

$$S \text{ is connected} \iff S \text{ is an interval}$$

Proposition

Let $f : X \to Y$ be a continuous map of topological spaces and X connected.
Then both $f(X)$ and the graph G_f of f are connected.

Proposition

Connectedness is a topological property.

Proposition

Suppose that $\{A_i : i \in I\}$ is an indexed family of connected subsets of a topological space X with $A_i \cap A_j \neq \varnothing$ for each pair $i, j \in I$. Then $\bigcup\limits_{i \in I} A_i$ is connected.

Proposition

Let $X \times Y$ be the topological product of spaces X, Y.

$$X \times Y \text{ is connected} \iff X \text{ and } Y \text{ are both connected}$$

Proposition

Suppose that A is a connected subset of a space X and that $A \subseteq B \subseteq \overline{A}$. Then B is connected

4.8 Path-connected spaces

Definition *Path-connected space*

For points x, y in a topological space X, a **path** in X from x to y is a continuous map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. We say that such a path joins x and y.
A topological space X is **path-connected** if any two points of X can be joined by a path in X.

Proposition

Any path-connected space X is connected.

Proposition

Suppose that $f, g : [0,1] \to X$ are paths in a space X from x to y and from y to z, respectively.
We can define a path in X from x to z as follows:

$$h(x) = \begin{cases} f(2t) & t \in [0, 1/2] \\ g(2t-1) & t \in [1/2, 1] \end{cases}$$

Proposition

A connected open subset U of $\mathbb{R}^n$ is path-connected.

5 Compact sets

Definition *Cover*

Suppose X is a set and $A \subseteq X$. A family $\{U_i : i \in I\}$ of subsets of X is called a **cover** for A if $A \subseteq \bigcup_{i \in I} U_i$

A **subcover** of a cover $\{U_i : i \in I\}$ for A is a subfamily $\{U_j : j \in J\}$ for some subset $J \subseteq I$ such that $\{U_j : j \in J\}$ is still a cover for A. We call it a **finite subcover** if J is finite. If $\mathcal{U} = \{U_i : i \in I\}$ is a cover for a subset A of a topological space X, and each U_i is open in X, then we call $\mathcal{U}$ an **open cover**.

Definition *Compact set*

A subset A of a topological space X is **compact** if every open cover for A has a finite subcover
A subset A of a topological space X is **relatively compact** in X if the closure $\overline{A}$ of A in X is compact.

Proposition

Let C be a compact subset of a Hausdorff space X. Then C is closed in X.

Proposition

If $f : X \to Y$ is a continuous map of topological spaces and X is compact then $f(X)$ is compact.

Proposition

Compactness is a topological property.

Proposition

Any closed subset C of a compact space X is compact.

Proposition

Let X, Y be topological spaces with topological product $X \times Y$.

$$X \times Y \text{ is compact} \iff X \text{ and } Y \text{ are both compact}$$

5.1 Compact sets in metric spaces

Proposition

Any compact subset C of a metric space (X, d) is bounded.

Proposition

Any continuous map from a compact space to a metric space is bounded.

5.2 Compact sets in Euclidean spaces

Proposition

Any closed bounded interval $[a, b]$ in $\mathbb{R}$ is compact.

Proposition *Heine-Borel Theorem*

Let $C \subseteq \mathbb{R}^n$.

$$C \text{ is closed and bounded} \implies C \text{ is compact}$$

Proposition

If $f : C \to \mathbb{R}$ is continuous and C is compact then f attains its bounds on C.
This means there exist $c_0 \in C$ and $c_1 \in C$ such that $f(c_0) = \inf f(C)$ and $f(c_1) = \sup f(C)$.
A continuous real-valued function on $[a, b]$ attains its bounds.

5.3 Uniform continuity

Definition *Uniform continuity*

A map $f : X \to Y$ of metric spaces (X, d_X) and (Y, d_Y) is **uniformly continuous** if:

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \quad d_X(x, a) < \delta \implies d_Y(f(x), f(a)) < \varepsilon \quad \text{for all } x, a \in X$$

Proposition

If $f : X \to Y$ is a continuous map of metric spaces and X is compact then f is uniformly continuous on X.

Proposition

If metrics d_1, d_2 for X are Lipschitz equivalent, then the identity map of (X, d_1) to (X, d_2) is uniformly continuous.

Proposition

Let X be a compact space and Y a Hausdorff space. Let $X : X \to Y$ be a continuous map.
If f is bijective, then f is a homeomorphism from X to Y.
If f is injective, then f is a homeomorphism from X to $f(X)$.

6 Sequences in metric spaces

6.1 Convergent sequences

Definition *Convergent sequence*

A sequence (x_n) in a metric space X **converges** to $x \in X$ if:

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } \quad n \geq N \implies x_n \in B_\varepsilon(x)$$

Proposition *Uniqueness of limits*

If a sequence (x_n) in a metric space X converges to both x and y in X, then $x = y$.

Definition *Cauchy sequence*

A sequence (x_n) in a metric space (X, d) is called a **Cauchy sequence** if:

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } \quad m, n \geq N \implies d(x_m, x_n) < \varepsilon$$

Proposition

Any convergent sequence in a metric space is a Cauchy sequence.

Proposition

Suppose that Y is a subset of a metric space X and (y_n) is a sequence in Y which converges to $a \in Y$. Then $a \in \overline{Y}$.
If Y is closed in X, then $a \in Y$.

Proposition

If a Cauchy sequence (x_n) in a metric space X has a subsequence converging to $x \in X$ then (x_n) converges to x.

6.2 Sequential compactness

Definition *Sequential compactness*

A metric space X or a subset $X \subseteq \mathbb{R}$ is **sequentially compact** if every sequence in X has at least one subsequence converging to a point of X. The empty set is also considered to be sequentially compact.
A non-empty subset A of a metric space is sequentially compact if this definition holds with the subspace metric d_A.

Proposition

Let $S \subseteq \mathbb{R}$.

$$S \text{ is closed and bounded} \iff S \text{ is compact} \iff S \text{ is sequentially compact}$$

Proposition

Let (X, d) be a metric space.

$$X \text{ is sequentially compact} \iff X \text{ is compact}$$

Proposition

Let (x_n) be a sequence in a metric space X and let $x \in X$.
If for all $\varepsilon > 0$, $B_\varepsilon(x)$ contains x_n for infinitely many values of n, then (x_n) has a subsequence converging to n.

Proposition

Suppose that a sequence (x_n) in a metric space X has no convergent subsequences.
Then for each $x \in X$ there exists $\varepsilon_x > 0$ such that B_{ε_x} contains x_n for only finitely many values of n.

Proposition

Any compact subset X of a metric space Y is sequentially compact.

***Definition** Lebesgue number*

Let $\mathcal{U}$ be any family of subsets of a metric space X covering a subset $A \subseteq X$. A **Lebesgue number** for $\mathcal{U}$ is a real number $\varepsilon > 0$ such that for all $a \in A$ the ball $B_\varepsilon(A)$ is contained in some single set from $\mathcal{U}$.

Proposition

Any open cover $\mathcal{U}$ of a sequentially compact metric space X has a Lebesgue number.

***Definition** ε-net*

Let $\varepsilon > 0$ be a real number and X a metric space. $N \subseteq X$ is an **ε-net** for X if the family $\{B_\varepsilon(x) : x \in N\}$ covers X.

Proposition

Let (X, d) be a sequentially compact metric space and let $\varepsilon > 0$. Then there exists a finite ε-net for X.

6.3 Uniform convergence

***Definition** Pointwise convergence*

A sequence (f_n) **converges pointwise** to f on D if for all $x \in D$, the sequence $(f_n(x))$ converges to $f(x)$.

***Definition** Uniform convergence*

A sequence (f_n) defined on a domain $D \subseteq \mathbb{R}$ **converges uniformly** on D to a function f if:

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that} \qquad |f_n(x) - f(x)| \leq \varepsilon \qquad \text{for all } x \in D,\ n \geq N$$

Proposition

f_n converges uniformly to f on D if $M_n = \sup_{x \in D} |f_n(x) - f(x)|$ exists for all sufficiently large n, and $\lim_{n \to \infty} M_n = 0$.

***Definition** Uniform Cauchy sequence*

A sequence (f_n) is said to be **uniformly Cauchy** on $D \subseteq \mathbb{R}$ if:

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that} \qquad |f_m(x) - f_n(x)| < \varepsilon \qquad \text{for all } m, n \geq N,\ x \in D$$

***Proposition** Cauchy's criterion for uniform convergence*

Let (f_n) be a sequence of real-valued functions defined on $D \subseteq \mathbb{R}$.

$$(f_n) \text{ converges uniformly on } D \iff (f_n) \text{ is uniformly Cauchy on } D$$

Proposition

If $f_n : (a, b) \to \mathbb{R}$ is continuous at $c \in (a, b)$ for all $n \in \mathbb{N}$ and $f_n \to f$ uniformly, then f is continuous at c.

Proposition

If $f_n : [a, b] \to \mathbb{R}$ is continuous for all $n \in \mathbb{N}$ and $(f_n) \to f$ uniformly on $[a, b]$, then f is continuous on (a, b).

Proposition

Suppose that the pointwise limit of a sequence (f_n) of continuous functions on $[a, b]$ is not continuous on $[a, b]$. Then the convergence is not uniform.

6.4 Complete spaces

***Definition** Complete space*

A metric space X is **complete** if every Cauchy sequence in X converges to a point in X.

Proposition

Suppose that X, Y are metric spaces and f is a bijective map such that f and f^{-1} are uniformly continuous.

$$X \text{ is complete} \iff Y \text{ is complete}$$

Proposition

Let X be a metric space and let d_1, d_2 be Lipschitz equivalent metrics.

$$(X, d_1) \text{ is complete} \iff (X, d_2) \text{ is complete}$$

Proposition

A complete subspace Y of a metric space X is closed in X.

Proposition

A closed subspace Y of a complete metric space X is complete.

Proposition

Any compact metric space X is complete.

Proposition

Let X, Y be metric spaces and with product $X \times Y$.

$$X \times Y \text{ is complete} \iff X \text{ and } Y \text{ are complete}$$

Proposition

Let $\mathcal{B}(D, X)$ be the space of bounded functions $D \to X$ with the sup metric d_∞.

$$(\mathcal{B}(D, X), d_\infty) \text{ is complete} \iff X \text{ is complete}$$

6.5 Fixed points and contractions

***Definition** Fixed point*

Given any set S and map $f : S \to S$, a **fixed point** of f is a point $p \in S$ such that $f(p) = p$

***Definition** Lipschitz condition*

Let α, K be constants with $\alpha > 0, K \geq 0$.
A function $f : D \to \mathbb{R}$ satisfies the **Lipschitz condition** of **order** α on D with **constant** K, if:

$$\text{for all } x, y \in D \quad |f(x) - f(y)| \leq K \cdot |x - y|^\alpha$$

Proposition

1. If f satisfies a Lipschitz condition of order $\alpha > 0$ on D then f is uniformly continuous on D.
2. If f satisfies a Lipschitz condition of order $\alpha > 1$ on $[a, b]$ then f is constant on $[a, b]$.
3. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) with $|f'(x)| \leq K$ for all $x \in [a, b]$, then f satisfies a Lipschitz condition of order 1 with constant K on $[a, b]$.

Proposition

Let D be an interval $[a, b]$, $(-\infty, b]$ or $[a, \infty)$, and $f : D \to D$ Lipschitz with $K < 1$ and $\alpha = 1$.
Then f has a unique fixed point $p \in D$.
Moreover, if x_1 is any point in D and $X_n = f(x_{n-1})$ for $n > 1$, then (x_n) converges to p.

Definition *Contraction*

Let (X, d) be a metric space. A function $f : X \to X$ is a **contraction** if there exists $0 \le K < 1$ such that:

$$d(f(x), f(y)) \le K \cdot d(x, y) \qquad \text{for all } x, y \in X$$

Proposition

Any contraction of a metric space X is uniformly continuous.

Proposition *Banach's fixed point theorem*

If $f : X \to X$ is a contraction of a complete metric space X, then f has a unique fixed point $p \in X$.

Construction of a fixed point

Let X be a complete metric space and $f : X \to X$ a contraction. Let $x_{n+1} := f(x_n)$.
The sequence (x_n) is Cauchy and therefore converges. The limit is the unique fixed point p.

Proposition

Let X be a complete metric space, $f : X \to X$ a contraction and p a fixed point of f.
Consider the sequence (x_n) defined by $x_{n+1} := f(x_n)$.

$$d(p, x_n) \le \frac{K^{n-1}}{1-K} d(x_2, x_1)$$

Proposition

Suppose that $K : [a, b] \times [a, b] \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous. Then the **Volterra equation**:

$$\phi(x) = f(x) + \int_a^k K(x, y)\phi(y)\, \mathrm{d}y$$

has a unique continuous solution ϕ on $[a, b]$.

Proposition *Cauchy-Picard theorem of differential equations*

Let $D = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$. Suppose that $f : D \to \mathbb{R}$ is continuous and there exists $K > 0$ such that:

$$|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2| \qquad \text{for all } (x, y_1), (x, y_2) \in D$$

Then on $I = [x_0 - c, x_0 + c]$, there exists a unique solution y of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \qquad y(x_0) = y_0$$

.

Index